

# UNIFORM SOBOLEV RESOLVENT ESTIMATES FOR THE LAPLACE-BELTRAMI OPERATOR ON COMPACT MANIFOLDS

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**ABSTRACT.** In this paper we continue the study on the resolvent estimates of the Laplace-Beltrami operator  $\Delta_g$  on a compact manifolds  $M$  with dimension  $n \geq 3$ . On the Sobolev line  $1/p - 1/q = 2/n$  we can prove that the resolvent  $(\Delta_g + \zeta)^{-1}$  is uniformly bounded from  $L^p$  to  $L^q$  when  $(p, q)$  are within the admissible range  $p \leq 2(n+1)/(n+3)$  and  $q \geq 2(n+1)/(n-1)$  and  $\zeta$  is outside a parabola opening to the right and a small disk centered at the origin. This naturally generalizes the previous results in [4] and [1] which addressed only the special case when  $p = 2n/(n+2)$ ,  $q = 2n/(n-2)$ . Using the shrinking spectral estimates between  $L^p$  and  $L^q$  we also show that when  $(p, q)$  are within the interior of the admissible range, one can obtain a logarithmic improvement over the parabolic region for resolvent estimates on manifolds equipped with Riemannian metric of non-positive sectional curvature, and a power improvement depending on the exponent  $(p, q)$  for flat torus. The latter therefore partially improves Shen's work in [6] on the  $L^p \rightarrow L^2$  uniform resolvent estimates on the torus. Similar to the case as proved in [1] when  $(p, q) = (2n/(n+2), 2n/(n-2))$ , the parabolic region is also optimal over the round sphere  $S^n$  when  $(p, q)$  are now in the admissible range. However, we may ask if the admissible range is sharp in the sense that it is the only possible range on the Sobolev line for which a compact manifold can have uniform resolvent estimate for  $\zeta$  being outside a parabola.

## 1 Introduction

Recall that in [4] (see also [1]) Dos Santos Ferreira, Kenig and Salo proved the following result concerning the resolvent estimates on a compact boundaryless Riemannian manifold:

**Theorem 1.1.** *Let  $(M, g)$  be a compact Riemannian manifold of dimension  $n \geq 3$ , and let  $\lambda, |\mu| \geq 1$ . Then there exists a uniform constant  $C > 0$  such that for all  $f \in C^\infty(M)$  we have the following resolvent estimate*

$$(1.1) \quad \|f\|_{L^{\frac{2n}{n-2}}(M)} \leq C \|(\Delta_g + (\lambda + i\mu)^2)f\|_{L^{\frac{2n}{n+2}}(M)}.$$

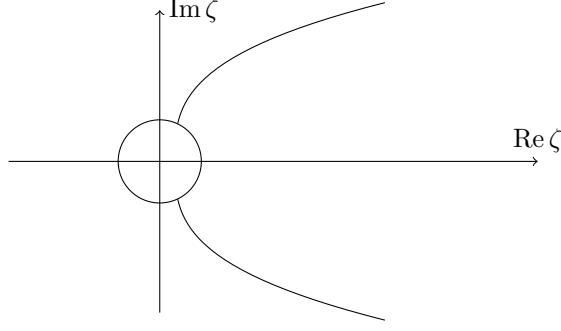
Notice that if we write  $\zeta = (\lambda + i\mu)^2$  then it is outside a small disk and a parabola opening to the right as in the following figure:

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Dos Santos Ferreira, Kenig and Salo [4] used explicit Hadamard parametrix construction to obtain the estimates above which is based on a classical representation of such parametrix in Bessel functions, see also [3]. Shortly after, Bougain, Sogge and us in [1] showed that estimate (1.1) is sharp on round sphere. They also used half-wave operator  $e^{it\sqrt{-\Delta_g}}$  and  $\cos t\sqrt{-\Delta_g}$  to prove the equivalence between any possible improvement over the parabola and shrinking spectral projection estimate of  $\sqrt{-\Delta_g}$ , and obtained some improvements on torus and non-positive curvature manifolds. In particular, using this technique they could obtain a shorter proof to Theorem 1.1.

The specific  $(p, q)$  pair appearing in (1.1) is at the intersection of the line of duality  $1/p + 1/q = 1$  and the Sobolev line  $1/p - 1/q = 2/n$ . Interestingly in the current paper we show that the line of duality does not play a significant role here, and the parabolic boundary of the region is essentially the result of the Sobolev line. More explicitly, we can prove that:

**Theorem 1.2.** *Let  $M$  be a compact Riemannian manifold of dimension  $n \geq 3$ . Then, if  $1/p - 1/q = 2/n$ , we have the following uniform resolvent estimates*

$$(1.2) \quad \|f\|_{L^q(M)} \leq C \|(\Delta_g + (\lambda + i\mu)^2)f\|_{L^p(M)}$$

*if  $p \leq 2(n+1)/(n+3)$  and  $q \geq 2(n+1)/(n-1)$ , and  $\lambda, |\mu| \geq 1$ . In particular, the constants  $C$  does not depend on  $\lambda, \mu$ .*

We follow our original way in [1] to prove this theorem by splitting the resolvent into short-time local part and long-time non-local remainder. The way we handle the local part is similar to, and motivated by the work in [4] through using the Carleson-Sjölin condition of an oscillatory term which we did not use in [1] since we concerned only the  $L^\infty$  norm of the kernel at that time. The main difference between our work and [4] is the way we handle the remainder term, whose  $L^p$  norm on the Sobolev line we are able to control by an argument using Sogge's spectral projection estimates.

The paper is organized as the following. As usual we interpret the resolvent  $(\Delta_g + (\lambda + i\mu)^2)^{-1}$  as a multiplier  $-(\tau^2 - (\lambda + i\mu)^2)^{-1}(P)$ , in which and following on,  $P$  denotes  $\sqrt{-\Delta_g}$ . We then calculate the Fourier transform of this multiplier function, and use the half-wave operator and Fourier inverse transform formula to write the resolvent as

$$(1.3) \quad \frac{\operatorname{sgn}\mu}{2i(\lambda + i\mu)} \int_{-\infty}^{\infty} e^{i(\operatorname{sgn}\mu)\lambda|t|} e^{-|\mu|t} e^{itP} dt = \frac{\operatorname{sgn}\mu}{i(\lambda + i\mu)} \int_0^{\infty} e^{i(\operatorname{sgn}\mu)\lambda t} e^{-|\mu|t} \cos tP dt.$$

Consequently we use a smooth function  $\rho(t)$  supported near  $t = 0$  to split the resolvent in (1.3) into local and non-local parts:

$$(1.4) \quad \mathfrak{S}_{loc}(P) = \frac{\operatorname{sgn} \mu}{i(\lambda + i\mu)} \int_0^\infty \rho(t) e^{i\lambda t} e^{-|\mu|t} \cos t P dt$$

and

$$(1.5) \quad r_{\lambda, \mu}(P) = \frac{\operatorname{sgn} \mu}{i(\lambda + i\mu)} \int_0^\infty (1 - \rho(t)) e^{i\lambda t} e^{-|\mu|t} \cos t P dt.$$

In Section 3 we study the non-local operator again in similar way as we did in [1], say breaking the spectrum of  $P$  into unit-length clusters and then estimate the  $(p, q)$  norm of the multiplier  $r_{\lambda, \mu}$  on each piece with the help of Sogge's spectral estimates in, say [10]. Our main tool is Lemma 3.2 which is a variant of Lemma 2.3 in [1]. The difference between this two lemmas is that instead of the standard  $TT^*$  argument we now consider the composition of spectral projections from  $L^p$  to  $L^2$  and from  $L^2$  to  $L^q$  in an asymmetric manner, which happens to behave well on the Sobolev line. By combining the results for local and non-local operator together, the proof to Theorem 1.2 is therefore completed.

Similar to the importance of Lemma 2.3 in [1], our Lemma 3.2 can immediately derive the following relation between shrinking spectral projection  $(p, q)$  estimates and the improved uniform resolvent estimates. Notice that unlike the case on the line of duality, we are not able to prove the exact equivalence between them:

**Theorem 1.3.** *Let  $M$  be a compact Riemannian manifold of dimension  $n \geq 3$ . Suppose that for  $2n(n+1)/(n^2+3n+4) \leq p \leq 2(n+1)/(n+3)$  we have a function  $0 < \varepsilon_p(\lambda) \leq 1$  decreasing monotonically to 0 as  $\lambda \rightarrow \infty$  and  $\varepsilon_p(2\lambda) \geq \varepsilon_p(\lambda)/2$ , for  $\lambda$  sufficiently large. Then if we have*

$$(1.6) \quad \left\| \sum_{|\lambda - \lambda_j| \leq \varepsilon_p(\lambda)} E_j f \right\|_{L^{p'}(M)} \leq C \varepsilon_p(\lambda) \lambda^{2\delta(p)} \|f\|_{L^p(M)}, \lambda \gg 1,$$

*we also have the following resolvent estimates for  $1/p - 1/q = 2/n, p \leq 2(n+1)/(n+3), q \geq 2(n+1)/(n-1)$ :*

$$(1.7) \quad \|f\|_{L^q(M)} \leq C \|(\Delta_g + (\lambda + i\mu)^2) f\|_{L^p(M)}, |\mu| \geq \max\{\varepsilon_p(\lambda), \varepsilon_{q'}(\lambda)\}, \lambda \gg 1.$$

With this theorem, we show in section 4 that the uniform resolvent estimates in Theorem 1.2 can be improved if the manifold  $M$  is equipped with a Riemannian metric with non-positive sectional curvature. More precisely we can prove the following theorem:

**Theorem 1.4.** *If  $M$  is a boundaryless Riemannian compact manifold with dimension  $\geq 3$  and of non-positive sectional curvature, then for  $1/p - 1/q = 2/n, p < 2(n+1)/(n+3), q > 2(n+1)/(n-1)$  we have the following uniform resolvent estimates*

$$(1.8) \quad \|f\|_{L^q(M)} \leq C \|(\Delta_g + (\lambda + i\mu)^2) f\|_{L^p(M)}$$

*if  $\lambda \gg 1$  and  $|\mu| > (\log(\lambda))^{-1}$ .*

The above theorem is an exemple in which one can get the same regional improvement for many  $(p, q)$  pairs on the Sobolev line, which is due to the slow growth of  $\log \lambda$  compared to any power of  $\lambda$ . In general the improvements may unsurprisingly depend on the

concrete value of  $(p, q)$  as we have seen in Theorem 1.3. In section 5 we prove the following theorem about the improved resolvent estimates on Torus  $\mathbb{T}^n$  for  $n \geq 3$  which serves as such an example:

**Theorem 1.5.** *Let  $\mathbb{T}^n$  denote the flat torus with  $n \geq 3$ . Then for a  $(p, q)$  pair satisfying  $1/p - 1/q = 2/n$  and  $p \leq 2(n+1)/(n+3)$ ,  $q \geq 2(n+1)/(n-1)$  there exists a function in  $p$ , which we denote by  $\varepsilon_n(p)$ , such that when  $1/p$  is ranging from  $(n+3)/2(n+1)$  to  $(n+2)/2n$  ( $\overline{AF}$  in figure 1) it increases from 0 to  $1/(n+1)$ , and symmetrically decreases from  $1/(n+1)$  to 0 when  $(n+2)/2n \leq 1/p \leq (n^2+3n+4)/2n(n+1)$  ( $\overline{FA'}$  in figure 1), and we have the following improved resolvent estimates*

$$(1.9) \quad \|f\|_{L^q(\mathbb{T}^n)} \leq C \|(\Delta_{\mathbb{T}^n} + (\lambda + i\mu)^2)f\|_{L^p(\mathbb{T}^n)}, \quad \lambda > 1, |\mu| \geq \lambda^{-\varepsilon(p)}.$$

The exact form for  $\varepsilon_n(p)$  is given in (5.11) when  $(1/p, 1/q)$  is below the line of duality.

Recall that in [6] Shen proved the following uniform resolvent estimates:

$$(1.10) \quad \|f\|_{L^q(\mathbb{T}^n)} \leq C \|(\Delta_{\mathbb{T}^n} + (\lambda + i\mu)^2)f\|_{L^2(\mathbb{T}^n)}, \quad \lambda, |\mu| \geq 1$$

for  $2 \leq q < 2(n-2)/(n-4)$  when  $n \geq 4$  and  $2 \leq q \leq \infty$  when  $n = 3$ . If we use Hölder inequality based on the fact that  $\mathbb{T}^n$  is compact and  $p < 2$  in (1.9), we can obtain similar  $L^2 \rightarrow L^q$  type resolvent estimates but with a much smaller  $q$ -range compared with (1.10). Our Theorem 1.5 on the other hand improves Shen's estimates in the aspect of allowing a smaller  $|\mu|$  comparable to certain negative power of  $\lambda$  for part of his exponent range.

The following figure 1 can be used by the interested readers to understand the admissible range of the  $(p, q)$  pair.  $\overline{AA'}$  will be from the non-local operator, which happens to be the global admissible range mentioned in Theorem 1.2. The Carleson-Sjölin argument used for local operator will give us the  $\overline{DO}$  and Young's inequality can give us segment  $\overline{EO}$ , therefore we can interpolate to obtain the segment  $\overline{CC'}$  with both end points removed as the admissible range for the local part. Since in general  $\overline{AA'} \subset \overline{CC'}$ , the resolvent estimate admissible range for a compact manifold is therefore, as far as we can prove, is constrained in  $\overline{AA'}$ . Notice that point  $F = (\frac{n+2}{2n}, \frac{n-2}{2n})$  is the  $(1/p, 1/q)$  pair considered in [4] and [1].

**Remark 1.6.** In [1] we showed that the parabolic region is sharp for round spheres with dimension  $\geq 3$  for pair  $(\frac{2n}{n+2}, \frac{2n}{n-2})$ . A simple duality argument and interpolation will show that this region is also sharp for our Theorem 1.2. We can somehow ask a question concerning the sharpness in a different manner: *Is the range  $\overline{AA'}$  sharp?* More precisely, is it possible to find a larger range than  $\overline{AA'}$  on the Sobolev line such that for a general compact manifold  $M$  we have uniform resolvent estimates as in Theorem 1.2. See Remark 3.3 for more discussion on this problem.

Throughout this paper  $\delta(p) = n|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}$  for  $1 \leq p \leq \infty$  and unless specified otherwise we generally assume  $1 \leq p \leq 2 \leq q \leq \infty$ .

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## 2 Local Operator

Our main theorem in this section is

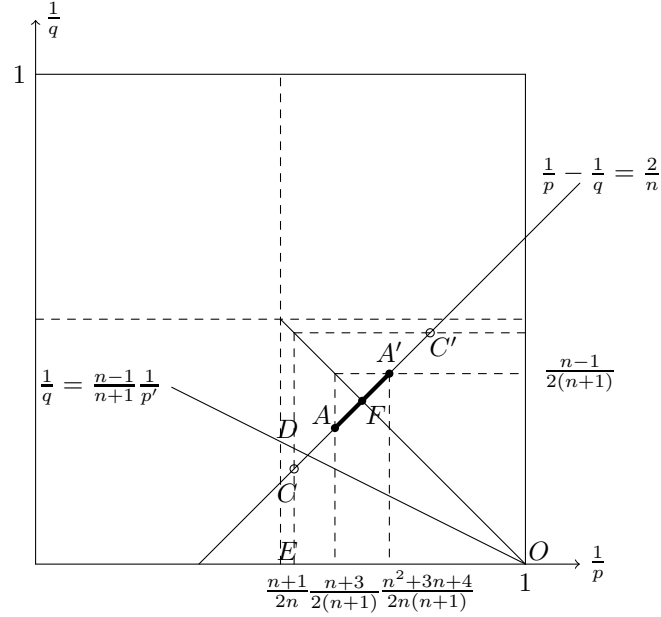


FIGURE 1. Admissible pairs

**Theorem 2.1.** *The local operator  $\mathfrak{S}_{loc}(P)$  is uniformly bounded for  $\lambda > 1$  and  $\mu \neq 0$  from  $L^p(M)$  to  $L^q(M)$  if  $1/p - 1/q = 2/n$  and  $p < 2n/(n+1), q > 2n/(n-1)$ , or more straightforward the segment  $\overline{CC'}$  in figure 1 with both end points removed.*

For simplicity we only prove the case when  $\mu > 1$ , the other case is symmetric. We firstly split the local resolvent into

$$\mathfrak{S}_{loc}(x, y) = \sum_{j=0}^{\infty} S_j(x, y)$$

in which

$$(2.1) \quad S_j f = \frac{1}{i(\lambda + i\mu)} \int_0^{\infty} \beta(\lambda 2^{-j} t) \rho(t) e^{i\lambda t - \mu t} \cos t P dt, \quad j \geq 1$$

and

$$(2.2) \quad S_0 f = \frac{1}{i(\lambda + i\mu)} \int_0^{\infty} (1 - \sum_{j=0}^{\infty} \beta(\lambda 2^{-j} t)) \rho(t) e^{i\lambda t - \mu t} \cos t P dt.$$

Here the function  $\beta \in C_0^{\infty}(\mathbb{R}^1)$  satisfies the following properties

$$(2.3) \quad \beta(t) = 0, t \notin [1/2, 2], |\beta(t)| \leq 1, \text{ and } \sum_{-\infty}^{\infty} \beta(2^{-j} t) = 1$$

Roughly speaking  $S_0$  is the worse part of the local operator as its time support is close to the singular point  $t = 0$ . So instead of the oscillatory integral technique we are going to use for  $S_j, j \geq 1$ , we take advantage of the  $O(\lambda^{-1})$  smallness of the time support to

directly estimate its kernel. Here we use a slightly simpler way to achieve this compared with the one used in [1]. In fact, we are going to prove that

**Lemma 2.2.** *The multiplier  $S_0(\tau)$  defined as*

$$(2.4) \quad S_0(\tau) = \frac{1}{i(\lambda + i\mu)} \int_0^\infty \tilde{\rho}(\lambda t) \rho(t) e^{i\lambda t} e^{-\mu t} \cos t\tau dt$$

*is a  $-2$  order symbol function with symbol norm independent from  $\lambda$  or  $\mu$ .*

*Proof.* Apparently due to the small  $t$  support in the integrand we know that when  $|\tau| < 1$  the integral, and similarly its  $\tau$  derivatives in (2.4) are uniformly bounded. Therefore we need only to prove that

$$(2.5) \quad \left| \frac{d^j}{d\tau^j} S_0(\tau) \right| \leq C_j \tau^{-2-j}, |\tau| \geq 1.$$

Let us prove firstly the case  $j = 0$  which may help the readers to understand how to handle the general case. Due to the fact that

$$(2.6) \quad \cos t\tau = \frac{1}{\tau} \frac{d}{dt} \sin t\tau, \quad \sin t\tau = -\frac{1}{\tau} \frac{d}{dt} \cos t\tau$$

we can do an integration by part twice and end up with some integrals boundary terms. Combining the fact that the integrand has a small  $t$  support  $t \leq 4\lambda^{-1}$ ,  $e^{-\mu t}\mu$  is interable uniformly in  $\mu$  and  $|\lambda + i\mu| \geq \lambda$  or  $\mu$  we immediately see that both the boundary terms and the integrals are uniformly bounded. This proves (2.5) when  $j = 0$ .

Now after taking  $j$  time  $\tau$  derivatives we should have

$$\frac{d^j}{d\tau^j} S_0 = \frac{\pm 1}{i(\lambda + i\mu)} \int \tilde{\rho}(\lambda t) \rho(t) e^{i\lambda t} e^{-\mu t} t^j \cos t\tau dt$$

and if  $j$  is odd we just need to replace  $\cos$  with  $\sin$ . But this is not important because we are then going to use (2.6) for  $j + 2$  times, and thanks to the appearance of  $t^j$  we do not need to worry the boundary condition until the last step. So now we are left with a bunch of integral terms and boundary terms. The boundary terms are similar as those ones from the  $j = 0$  because there must be  $j$  times integration by parts being used to consume  $t^j$ . Therefore we need only to discuss those integrals. To simply our argument we can classify them into three classes. The first ones are those terms containing  $\mu^{k+2}t^k$  for some  $k \geq 0$ ; the second ones are those terms containing  $\mu^{k+1}t^k$  with  $k \geq 0$ ; and all the others.

For the first ones, we know that there must be no derivative hitting the  $\tilde{\rho}(\lambda t)e^{i\lambda t}$  factor, so we use a scaling  $t \rightarrow \mu^{-1}t$  to bound these integrals. Here we also used the property that  $|\lambda + i\mu| > \mu$ . For some integral in the second class, there might be one and at most one derivative hitting  $\tilde{\rho}(\lambda t)e^{i\lambda t}$ , so we again use a scaling and the fact that  $|\lambda + i\mu| > \lambda$  to control this integral. The other terms are more easily to be handled since  $e^{-\mu t}(\mu t)^k, k \geq 0$  are uniformly bounded therefore can be ignored. We then use the small  $t$  support  $t \leq \lambda^{-1}$  to control all the large derivatives coming from  $\tilde{\rho}(\lambda t)e^{i\lambda t}$  if there are any. So the proof is completed. □

With the aid of this lemma, we know that  $S_0(P)$  is therefore a  $-2$  order pseudodifferential operator, and in particular with symbol norm uniformly bounded from  $\lambda$  or  $\mu$ . This then leads us to the following kernel estimate (see for example [11]) due to the fact that  $n \geq 3$ :

$$|S_0(x, y)| \leq C d_g(x, y)^{2-n}.$$

By the Hardy-Littlewood-Sobolev inequality,  $S_0$  is a  $L^p \rightarrow L^q$  bounded operator on Sobolev line.

To deal with  $S_j, j \geq 1$ , we need the following version of Proposition 2.4 in [1]

**Proposition 2.3.** *Let  $n \geq 2$  and assume that  $a \in C^\infty(\mathbb{R}_+)$  satisfies the Mihlin-type condition that for each  $j = 0, 1, \dots, 2$*

$$(2.7) \quad \left| \frac{d^j}{ds^j} a(s) \right| \leq A_j s^{-j}, s > 0$$

*Then there are constants  $B, B_j$ , which depend only on the size of finitely many of the constants  $A_j$  so that for every  $\omega \in \mathbb{C}$  such that  $\text{Im } \omega \neq 0$  and  $1/4 < |x| < 4$  we will have*

$$(2.8) \quad \int_{\mathbb{R}^n} \frac{a(|\xi|) e^{ix \cdot \xi}}{|\xi| - \omega} d\xi = |x|^{1-n} a_{1,\omega}(x) + \sum_{\pm} e^{\pm i(\text{Re } \omega)|x|} |\text{Re } \omega|^{-\frac{n-1}{2}} |x|^{-\frac{n-1}{2}} a_{2,\omega}^{\pm}(|x|),$$

*in which  $|a_{1,\omega}(x)| = B$  is a bounded smooth function,  $|\frac{d^j}{ds^j} a_{2,\omega}^{\pm}(s)| \leq B_j s^{-j}$  therefore  $a_{2,\omega}^{\pm}(|x|)$  have bounded derivatives under our assumption on  $|x|$ .*

*Proof.* If  $\text{Re } \omega = 0$ , then we take  $\xi \rightarrow \varepsilon \xi$  scaling to prove (2.8) since now the oscillatory integral is the Fourier transform of a  $-1$  order Mihlin-type symbol function. We then assume from now on that  $\text{Re } \omega \neq 0$ , which allows us to take the scaling  $\xi \rightarrow \text{Re } \omega \xi$ . Notice that if  $|\text{Im } \omega / \text{Re } \omega| \geq 1$  or  $|\xi| \notin (1/4, 4)$  then we again have a  $-1$  order Mihlin-type symbol function with symbol norms uniformly bounded, so we need only to consider the following integral

$$(2.9) \quad |\text{Re } \omega|^{n-1} \int \frac{\beta(|\xi|) a(\text{Re } \omega |\xi|) e^{i \text{Re } \omega x \cdot \xi}}{|\xi| - 1 + i\varepsilon}, \quad 0 < |\varepsilon| < 1$$

in which  $\beta(r)$  is a smooth function supported in  $(1/4, 4)$  and equal to 1 in  $(1/2, 2)$ . For simplicity we write  $\beta(|\xi|) a(\text{Re } \omega |\xi|)$  as  $\alpha(|\xi|)$ . Now when  $|\text{Re } \omega| \cdot |x| < 1$ , we can use the property that function  $\alpha(|\xi|) e^{i \text{Re } \omega x \cdot \xi}$  has bounded  $\xi$  derivatives to show that the integral in (2.9) will be uniformly bounded under such assumption. Therefore we need only to consider the case when  $|\text{Re } \omega| |x| \geq 1$ .

Now after using polar coordinates and especially the Fourier transform of the sphere, we should obtain the following integral

$$(2.10) \quad \left( \left| \frac{\text{Re } \omega}{x} \right| \right)^{\frac{n-1}{2}} \int_{\frac{1}{4}}^4 \frac{\alpha(r) c_{\pm}(r |\text{Re } \omega| |x|)}{r - 1 + i\varepsilon} r^{\frac{n-1}{2}} e^{\pm i r \text{Re } \omega |x|} dr, \quad 0 < |\varepsilon| < 1$$

in which  $c_{\pm}(r)$  satisfies that

$$\left| \frac{d^j}{dr^j} c_{\pm}(r) \right| \geq r^{-j}, r \geq 1/4.$$

Now let us recall the following lemma which was also proved in [1]:

**Lemma 2.4.**

$$(2.11) \quad \int \frac{e^{-irt}}{r-1+i\varepsilon} = 2\pi i H(\varepsilon t) e^{-it} e^{-|\varepsilon t|},$$

in which  $H(t)$  is the Heaviside function.

Now we can regard the integral in (2.10) as a convolution between the function in (2.11) and  $b_{\pm}(t, |x|)$ , in which  $b_{\pm}(t, |x|)$  denotes the  $r$  Fourier transform of function  $\alpha(r)c_{\pm}(r\text{Re } \omega|x|)r^{\frac{n-1}{2}}$ . Notice that in particular we have

$$(2.12) \quad |D_x^\gamma D_t^\alpha b_{\pm}(t, |x|)| \leq C_{N,\gamma,\alpha} (1+t)^{-N}$$

due to the support of  $|x|$  and the assumption that  $|\text{Re } \omega| \cdot |x| > 1$ . Without loss of generality we assume  $\text{Re } \omega > 0$  and  $\varepsilon > 0$  and the other cases can be handled similarly. Under such assumption the convolution should be:

$$(2.13) \quad 2\pi i e^{\mp i \text{Re } \omega |x|} (e^{-\text{Re } \omega \varepsilon |x|} \int_{-\infty}^{\text{Re } \omega |x|} e^{it} e^{\varepsilon t} b_{\pm}(t, |x|) dt).$$

Therefore the proof to the proposition will be completed as long as we can show that the function in the parentheses has uniformly bounded  $x$  derivatives.

In fact, we should have

$$\begin{aligned} & \partial_x^\gamma (e^{-\text{Re } \omega \varepsilon |x|} \int_{-\infty}^{\text{Re } \omega |x|} e^{it} e^{\varepsilon t} b_{\pm}(t, |x|) dt) \\ &= \sum_{|j|+|k|=|\gamma|-1} C_{j,k} \partial_x^j (\text{Re } \omega \varepsilon |x| e^{-\text{Re } \omega \varepsilon |x|}) \partial_x^k (e^{i \text{Re } \omega |x|} b_{\pm}(\text{Re } \omega |x|, |x|)) \\ & \quad + \sum_{j+k=\gamma} C_{j,k} \partial_x^j (e^{i \text{Re } \omega \varepsilon |x|}) \int_{-\infty}^{\text{Re } \omega |x|} e^{it} e^{-\varepsilon t} \partial_x^k (b_{\pm}(t, |x|)) dt \\ &= I + II. \end{aligned}$$

By the fact that  $\varepsilon < 1$ ,  $|x| \approx 1$  and (2.12) we immediately see that  $I$  is uniformly bounded. Now for  $II$ , after a  $t \rightarrow -t + \text{Re } \omega$  variable substitution the terms in it are majorized by the following integral

$$(2.14) \quad (\text{Re } \omega \varepsilon)^j \int_0^\infty e^{-\varepsilon t} |b_{\pm}^{(k)}(-t + \text{Re } \omega, |x|)| dt \leq C_{j,k,N} \int_0^\infty (\varepsilon \text{Re } \omega)^j (1+\varepsilon t)^{-j} (1+|t - \text{Re } \omega|)^{-N} dt.$$

By arguing the ratio between  $t$  and  $\text{Re } \omega$  we can immediately show that  $II$  is also uniformly bounded. So the proof is completed.  $\square$

Now, let  $\varepsilon = \frac{2^j}{\lambda}$ , so the kernel of  $S_j$  is

$$(2.15) \quad S_j(x, y) = \frac{1}{i(\lambda + i\mu)} \int_0^\infty \beta\left(\frac{t}{\varepsilon}\right) \rho(t) e^{\lambda t} e^{-\mu t} \cos t P(x, y) dt.$$

We should notice that due to the finite propagation speed of the wave operator  $\cos t P$  the kernels  $S_j(x, y)$  are actually supported in the region where  $|d_g(x, y)/\varepsilon| < 2$ , so we have

$$(2.16) \quad S_j(x, y) = \rho(x, y, \varepsilon) S_j(x, y)$$



in which we recall that  $\rho$  is a smooth bump function supported when  $d_g(x, y) \leq 4\varepsilon$ . By such consideration we can therefore restrict the support of all the following operators to such a small region.

By Euler formula, in geodesic coordinates we have

$$(2.17) \quad \cos tP(x, y) = \sum_{\pm} \int_{\mathbb{R}^n} e^{i\kappa(x, y) \cdot \xi} e^{i\pm t|\xi|} \alpha_{\pm}(t, x, y, |\xi|) d\xi + Q(t, x, y)$$

in which  $Q(t, x, y)$  is a smooth function with compact support in  $t, x, y$ . So if we replace the operator  $\cos tP$  in (2.15) we have a operator with  $(p, q)$  norm equal to  $O(\lambda^{-1}\varepsilon)$ . Notice that due to the  $t$  support in (2.15) we would have that there are only  $O(\log_2 \lambda)$  many  $S_j$  not vanishing. So summing over so many  $j$  will give us a  $(p, q)$  bounded operator over the Sobolev line immediately. Notice that by (2.16) we can assume that both the Fourier integral and  $Q(t, x, y)$  are supported when  $d_g(x, y)/\varepsilon < 4$ .

So by abusing language a little bit, we can replace the wave operator  $\cos tP$  with the Fourier integral representation in (2.17). Now take a  $(t, \xi) \rightarrow (\varepsilon t, \xi/\varepsilon)$  scaling in (2.15) we then should obtain

$$(2.18) \quad S_j^{\pm}(x, y) = \frac{\varepsilon^{1-n}}{i(\lambda + i\mu)} \int_0^{\infty} \int_{\mathbb{R}^n} \beta(t) \rho(\varepsilon t) e^{i\lambda \varepsilon t} e^{-\mu \varepsilon t} \alpha_{\pm}(\varepsilon t, x, y, \frac{|\xi|}{\varepsilon}) e^{i\frac{\kappa(x, y)}{\varepsilon} \cdot \xi \pm t|\xi|} d\xi dt$$

for  $d_g(x, y)/\varepsilon < 4$ . However, when  $d_g(x, y)/\varepsilon \leq 1/4$ , an integration by parts argument with respect to  $\xi$  would show that

$$|S_j^{\pm}(x, y)| \leq \varepsilon^{1-n} \lambda^{-1} \leq d_g(x, y)^{2-n} 2^{-j},$$

which is a  $(p, q)$  bounded operator over the Sobolev line after summation over  $j$ . So we are reduced to considering the operator  $K_j^{\pm} = \beta(\frac{d_g(x, y)}{2\varepsilon}) S_j^{\pm}(x, y)$  in which  $\beta(r)$  is supported when  $r \in (1/2, 2)$ .

We then proceed as in [1]. More specifically, let  $a_{\varepsilon}^{\pm}(\tau, x, y, |\xi|)$  denote the inverse Fourier transform of

$$t \rightarrow \beta(t) \rho(\varepsilon t) \alpha_{\pm}(\varepsilon t, x, y, |\xi|/\varepsilon),$$

in which

$$(2.19) \quad |D_{\xi}^{\gamma} a_{\varepsilon}^{\pm}(\tau, x, y, |\xi|)| \leq C_{N, \gamma} (1 + \tau)^{-N} |\xi|^{-|\tau|}.$$

Then after the Fourier transform in (2.18) we would have

$$(2.20) \quad K_j^{\pm}(x, y) = \varepsilon^{2-n} \int \int_{\mathbb{R}^n} e^{i\frac{\kappa(x, y)}{\varepsilon} \cdot \xi} \frac{a_{\varepsilon}^{\pm}(\tau, x, y, |\xi|)}{(\pm|\xi| - \tau - \varepsilon\lambda - i\varepsilon\mu)(\pm|\xi| - \tau + \varepsilon\lambda + i\varepsilon\mu)} d\xi d\tau.$$

Now if we split the fraction in the integrand as following

$$\begin{aligned} & \frac{1}{(\pm|\xi| - \tau - \varepsilon\lambda - i\varepsilon\mu)(\pm|\xi| - \tau + \varepsilon\lambda + i\varepsilon\mu)} \\ &= \frac{1}{2(\varepsilon\lambda + i\varepsilon\mu)} \left( \frac{1}{\pm|\xi| - \tau - \varepsilon\lambda - i\varepsilon\mu} - \frac{1}{\pm|\xi| - \tau + \varepsilon\lambda + i\varepsilon\mu} \right) \end{aligned}$$

then we would have, after applying Proposition 2.3,  
(2.21)

$$K_j^\pm(x, y) = 2^{-j} \varepsilon^{2-n} a_{1,\omega}(x, y) + 2^{-j} \varepsilon^{2-n} \sum_{\pm} \int e^{\pm i(\tau \pm \varepsilon \lambda) \frac{d_g(x, y)}{\varepsilon}} \cdot |\tau \pm \varepsilon \lambda|^{\frac{n-1}{2}} a_\varepsilon(\tau, x, y) d\tau$$

in which  $a_{1,\omega}(x, y)$  is a uniformly bounded smooth function with support when  $d_g(x, y)/\varepsilon \in (1/4, 4)$ , and  $a_\varepsilon(\tau, x, y)$  is virtually the function  $a_{2,\omega}$  in (2.8) and it particular, it has the following properties

$$(2.22) \quad |\partial_{x,y}^\gamma a_\varepsilon(\tau, x, y)| \leq C_{\gamma,N} (1 + \tau)^{-N} \varepsilon^{-|\gamma|}$$

also due to the fact that  $d_g(x, y)/\varepsilon \approx 1$ .

Now for the  $2^{-j} \varepsilon^{2-n} a_{1,\omega}(x, y)$  piece, simple calculation shows that its  $L^1 \rightarrow L^\infty$  norm is  $2^{-j} \varepsilon^{2-n}$  and the  $L^p \rightarrow L^p$  norm is  $2^{-j} \varepsilon^2, 1 \leq p \leq \infty$ . So interpolation shows that these operators sum up to a  $(p, q)$  bounded operator on the Sobolev line. Therefore it reduces to analyzing the second operator in (2.21). After carrying out the  $\tau$  integral, we immediately see that we are further reduced, without loss of generality, to estimating the  $(p, q)$  bounds over the Sobolev line for the following operators

$$(2.23) \quad T_j(x, y) = 2^{-j} \varepsilon^{2-n} 2^j \frac{n-1}{2} e^{i\lambda d_g(x, y)} a_\varepsilon(x, y) = \lambda^{\frac{n-3}{2}} \varepsilon^{-\frac{n-1}{2}} e^{i\lambda d_g(x, y)} a_\varepsilon(x, y)$$

in which the smooth function  $a_\varepsilon(x, y)$  is supported when  $d_g(x, y)/\varepsilon \in (1/4, 4)$ , and

$$(2.24) \quad |\partial_{x,y}^\gamma a_\varepsilon(x, y)| \leq C_\gamma \varepsilon^{-|\gamma|}.$$

Now the rest part will be standard procedure as in [10] since we know the function  $d_g(x, y)$  satisfies the Carleson-Sjölin condition, and this was also done similarly in [4]. See [3] also for the Euclidean case. For the reader's convenience we state it briefly as following. At first, (2.22) reminds us that if we scale  $x, y$  back to unit-length by  $(x, y) \rightarrow (\varepsilon x, \varepsilon y)$  we should have

$$(2.25) \quad |\partial_{x,y}^\gamma a_\varepsilon(\varepsilon x, \varepsilon y)| \leq C_\gamma.$$

Also the phase function now is  $\lambda d_g(\varepsilon x, \varepsilon y) = 2^j d_g(\varepsilon x, \varepsilon y)/\varepsilon$  and  $d_g(\varepsilon x, \varepsilon y)/\varepsilon$  will satisfy Carleson-Sjölin condition such that the hypersurface  $\nabla_x d_g(\varepsilon x, \varepsilon y_0)/\varepsilon$  will have curvature bounded away from zero. So, if  $f(y)$  is a test function, then we should have, if

$$(2.26) \quad \frac{1}{q} = \frac{n-1}{n+1} \left(1 - \frac{1}{p}\right), \quad 1 \leq p \leq 2,$$

that

$$\begin{aligned} & \|T_j(x, y) f(y)\|_q \\ &= \lambda^{\frac{n-3}{2}} \varepsilon^{-\frac{n-1}{2}} \left( \int \left| \int e^{-i\lambda d_g(x, y)} a_\varepsilon(x, y) f(y) dy \right|^p dx \right)^{\frac{1}{q}} \\ (2.27) \quad &= \lambda^{\frac{n-3}{2}} \varepsilon^{-\frac{n-1}{2}} \varepsilon^{n+\frac{n}{q}} \left( \int \left| \int e^{-i2^j d_g(\varepsilon x, \varepsilon y)/\varepsilon} a_\varepsilon(\varepsilon x, \varepsilon y) f(\varepsilon y) dy \right|^p dx \right)^{\frac{1}{q}} \\ &\lesssim \lambda^{\frac{n-3}{2}} \varepsilon^{-\frac{n-1}{2}} \varepsilon^{n+\frac{n}{q}} 2^{\frac{-jn}{q}} \varepsilon^{-\frac{n}{p}} \|f\|_p \\ &= \lambda^{-2+n(\frac{1}{p}-\frac{1}{q})} 2^{j(\frac{n+1}{2}-\frac{n}{p})} \|f\|_p. \end{aligned}$$

Here the inequality is due to the standard  $n \times n$  Carleson-Sjölin estimate in [10]. On the other hand, if we apply Young's inequality to the kernel  $T_j(x, y)$  as in (2.23), we also see that the kernel on the line  $(1/p, 0)$  will be  $L^p \rightarrow L^q$  bounded by the same norm as

in (2.27) with corresponding exponent. Now a simple interpolation shows that our local operator is  $L^p \rightarrow L^q$  bounded when  $(1/p, 1/q)$  are on  $\overline{CC'}$ , with end points removed.

**Remark 2.5.** Through Lemma 2.2 and the oscillatory integral (2.20) the reader may have noticed that our local operator  $\mathfrak{S}_{loc}(P)$  bears great similarity compared with the classical Hadamard parametrix to  $(\Delta_g + \zeta)^{-1}$  used in [4]. This is in fact not surprising since they are essentially the same operator expressed in different ways. The reader may also compare this result with Theorem 2.2 in [3], which is the Euclidean case when we replace the local resolvent by the resolvent  $(\Delta_{\mathbb{R}^n} + \zeta)^{-1}$  to see the similarity between the local operator and its Euclidean counterpart.

Also when  $n = 2$ , due to the fact that a  $-n$  order pseudodifferential operator in  $\mathbb{R}^n$  has kernel bounded by  $\log|x - y|^{-1}$  (see for example [13]), the readers can easily prove the local operator  $\mathfrak{S}_{loc}(P)$  is now bounded from  $H^1(M)$ , the Hardy space whose definition we adopt from [12], to  $L^\infty(M)$ .

### 3 Non-local Operator

Now let us deal with the non-local operator

$$(3.1) \quad r_{\lambda, \mu}(P) = \frac{1}{i(\lambda + i\mu)} \int_0^\infty (1 - \rho(t)) e^{i\lambda t} e^{-\mu t} \cos tP dt$$

in which we assume that  $\mu \geq 1$ . This operator is more easier than its local counterpart to handle and we prove:

**Theorem 3.1.** *The non-local operator  $r_{\lambda, \mu}(P)$  defined in (3.1) is a uniformly bounded operator from  $L^p(M)$  to  $L^q(M)$  if  $1/p - 1/q = 2/n$  and especially when  $p \leq 2(n+1)/n+3$ ,  $q \geq 2(n+1)/(n-1)$ . Notice that this is the segment  $\overline{AA'}$  given in figure 1.*

Similar to [1], the proof is based on the following lemma:

**Lemma 3.2.** *Given a fixed compact Riemannian manifold of dimension  $n \geq 3$  there is a constant  $C$  so that whenever  $\alpha \in C(\mathbb{R}_+)$  and let*

$$\alpha_k(P)f = \sum_{\lambda_j \in [k-1, k)} \alpha(\lambda_j) E_j f, k = 1, 2, \dots$$

*then we should have*

$$(3.2) \quad \|\alpha_k(P)f\|_{L^q(M)} \leq Ck \left( \sup_{\tau \in [k-1, k)} |\alpha(\tau)| \right) \|f\|_{L^p(M)}$$

*if  $1/p - 1/q = 2/n$  and  $p \leq 2(n+1)/(n+3)$ ,  $q \geq 2(n+1)/(n-1)$ .*

To prove the lemma firstly, let us recall the following theorem in [10] and [8]:

**Theorem.** *If  $\chi_k$  denotes the spectral cluster projection of the operator  $P = \sqrt{-\Delta_g}$ , namely  $\chi_k f = \sum_{\lambda_j \in [k-1, k)} E_j f$ , then we have the following estimates*

$$(3.3) \quad \|\chi_\lambda f\|_{L^2(M)} \leq C(1 + \lambda)^{\delta(p)} \|f\|_{L^p(M)}, \quad 1 \leq p \leq \frac{2(n+1)}{n+3},$$

*and*

$$(3.4) \quad \|\chi_\lambda\|_{L^q(M)} \leq C(1 + \lambda)^{\delta(q)} \|f\|_{L^2(M)}, \quad \frac{2(n+1)}{n-1} \leq q \leq \infty$$

in which  $\delta(p) = n|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}$ .

Now since  $\alpha_k(P) = \chi_k \alpha_k(P) \chi_k$ , we have

$$\begin{aligned} \|\alpha_k(P)f\|_{L^q(M)} &\lesssim k^{\delta(q)} \|\alpha_k(P)\chi_k f\|_{L^2(M)} \\ &\lesssim k^{\delta(q)} \left( \sup_{\tau \in [k-1, k)} |\alpha(\tau)| \right) \|\chi_k f\|_{L^2(M)} \\ &\lesssim k^{\delta(q)+\delta(p)} \left( \sup_{\tau \in [k-1, k)} |\alpha(\tau)| \right) \|f\|_{L^p(M)}. \end{aligned}$$

On the Sobolev line we happen to have  $\delta(p) + \delta(q) = 1$ . This completes the proof to the lemma. Now to prove the Theorem 3.1, we just need to notice that under our assumption  $\mu \geq 1$  we have the following estimate:

$$(3.5) \quad |r_{\lambda, \mu}(\tau)| \leq C_N \lambda^{-1} ((1 + |\lambda - \tau|)^{-N} + (1 + |\lambda + \tau|)^{-N}), \quad \lambda, \mu \geq 1.$$

So now the Lemma (3.2) comes into application immediately if we take  $N = 3$  as following:

$$(3.6) \quad \|r_{\lambda, \mu}(P)\|_{L^p \rightarrow L^q} \leq C \sum_{k=1}^{\infty} k \lambda^{-1} (1 + |\lambda - k|)^{-3} \leq C'$$

In the previous section we have already proved that when  $(1/p, 1/q)$  are in the admissible range the local operator is uniformly bounded between  $L^p(M)$  and  $L^q(M)$  with only the requirement that  $\mu \neq 0$ . Theorem 1.2 is therefore proved.

*Proof of Theorem 1.3.* With the help of Lemma (3.2) we can prove Theorem 1.3 now. As we have seen in Section 2 that if we replace the resolvent in (1.7) by the local operator  $\mathfrak{S}_{loc}(P)$  then the estimates hold automatically, therefore we need only to prove that the non-local operator  $r_{\lambda, \mu}(P)$  satisfies the same estimates

$$(3.7) \quad \|r_{\lambda, \mu} f\|_{L^q(M)} \leq C \|f\|_{L^p(M)},$$

under the assumption of (1.6) and in particular when  $|\mu| \lesssim 1$ . This is again an application of Lemma 3.2 with finer arguments.

Firstly, integration by parts in (3.1) shows that we have

$$(3.8) \quad |r_{\lambda, \mu}(\tau)| \leq C \lambda^{-1} [(1 + |\lambda - \tau|)^{-N} + |\mu|^{-1} (1 + |\mu|^{-1} |\lambda - \tau|)^{-N}], \quad N = 0, 1, \dots$$

Now, assume we have a positive number  $\alpha$  such that  $\varepsilon_p(\lambda) \leq \alpha \leq 1$  in which  $p$  is any number within the range mentioned in Theorem 1.3. Notice that if we partition the interval  $[\lambda - \alpha, \lambda + \alpha]$  evenly into  $O(\alpha/\varepsilon_p(\lambda))$  pieces of small interval of length  $\varepsilon_p(\lambda)$ , then by Minkowski inequality,  $L^2$  orthogonality and our special requirement on  $\varepsilon_p(\lambda)$  that  $\varepsilon_p(\lambda) \approx \varepsilon_p(\lambda + 1)$ , we immediately have

$$(3.9) \quad \left\| \sum_{|\lambda - \lambda_j| \leq \alpha} E_j f \right\|_{L^{p'}(M)} \leq C \lambda^{2\delta(p)} \alpha \|f\|_{L^p(M)}, \quad \varepsilon_p(\lambda) \leq \alpha \leq 1$$

In particular the constant  $C$  is uniform. This amounts to say that if the shrinking spectral estimates hold for a smaller spectral cluster, they must hold as well for any larger cluster

(but still shorter than unit length, of course). Now using  $L^2$  orthogonality again, if we have  $\lambda' \in [\lambda - 1, \lambda + 1]$ , we have

$$(3.10) \quad \left\| \sum_{|\lambda' - \lambda_j| \leq \alpha} r_{\lambda, \mu}(\lambda_j) E_j f \right\|_{L^{p'}(M)} \leq C \alpha \lambda^{2\delta(p)} \left( \sup_{|\lambda' - \tau| \leq \alpha} |r_{\lambda, \mu}(\tau)| \right) \|f\|_{L^p(M)}, \quad \varepsilon_p(\lambda) \leq \alpha \leq 1.$$

So if we use  $TT^*$  arguments to break (3.9) into  $L^p(M) \rightarrow L^2(M)$  and  $L^2(M) \rightarrow L^q(M)$  then compose them together while choosing  $\alpha = |\mu|$ , we would have

$$(3.11) \quad \left\| \sum_{|\lambda' - \lambda_j| \leq |\mu|} r_{\lambda, \mu}(\lambda_j) E_j f \right\|_{L^q(M)} \leq C |\mu| \lambda \left( \sup_{|\lambda' - \tau| \leq |\mu|} |r_{\lambda, \mu}(\tau)| \right) \|f\|_{L^p(M)},$$

in which  $|\mu| \geq \max \{ \varepsilon_p(\lambda), \varepsilon_{q'}(\lambda) \}$ .

Now for the non-local operator we have

$$(3.12) \quad \|r_{\lambda, \mu} f\|_{L^q(M)} \leq \left\| \sum_{|\lambda - \lambda_j| > 1} r_{\lambda, \mu}(\lambda_j) E_j f \right\|_{L^q(M)} + \left\| \sum_{|\lambda - \lambda_j| \leq 1} r_{\lambda, \mu}(\lambda_j) E_j f \right\|_{L^q(M)}.$$

For the first summand, we simply use the Lemma 3.2 to control it. For the second summand, we can evenly partition  $[\lambda - 1, \lambda + 1]$  into small intervals  $I_k$  of length comparable to  $|\mu|$  as:

$$(3.13) \quad I_k = \{ \tau \in [\lambda - 1, \lambda + 1] : k|\mu| \leq |\tau - \lambda| \leq (k + 1)|\mu| \}, \quad k = 0, 1, \dots$$

then take sum using (3.8) and (3.11) to see

$$(3.14) \quad \begin{aligned} & \left\| \sum_{|\lambda - \lambda_j| \leq 1} r_{\lambda, \mu}(\lambda_j) E_j f \right\|_{L^q(M)} \\ & \leq C \lambda^{-1} |\mu| \sum_k \left[ (1 + |\mu|k)^{-N} + |\mu|^{-1} (1 + k)^{-N} \right] \|f\|_{L^p(M)} \\ & \leq C \|f\|_{L^p(M)}. \end{aligned}$$

□

**Remark 3.3.** When points  $(1/p, 1/q)$  are off the admissible range  $\overline{AA'}$  in figure 1, the above technique we used to show the boundedness of the non-local operator will not work. In fact, when  $2(n+1)/(n+3) < p \leq 2$  we have the following spectral projection estimates

$$(3.15) \quad \|\chi_\lambda f\|_{L^2(M)} \leq C(1 + \lambda)^{\frac{(n-1)(2-p)}{4p}} \|f\|_{L^p(M)}.$$

Now assume that we have some  $(1/p, 1/q)$  being on the Sobolev line but on the left hand side of point  $A$ , then after the  $L^p \rightarrow L^2$  and  $L^2 \rightarrow L^q$  composition we have

$$(3.16) \quad \|\chi_\lambda f\|_{L^q(M)} \leq C(1 + \lambda)^{\frac{(n-1)(2-p)}{4p} + \delta(1/(\frac{1}{p} - \frac{2}{n}))} \|f\|_{L^p(M)}$$

and an easy calculation shows that now the power of  $\lambda$  is within  $(1, 3/2]$ , which is not sufficient to control the non-local operator as the readers have seen. However, unlike the  $L^p \rightarrow L^{p'}$  estimate, a general  $L^p \rightarrow L^q$  estimate for  $p$  and  $q$  being off the line of duality obtained by composing two projections together may not be sharp. Therefore further improvement on the admissible range may still be possible. In fact, if we assume  $M = S^3$  and  $\mu = 1$ , then at the end point  $p = 4/3$  and  $q = 12$  the resolvent estimates now read as

$$(3.17) \quad \|f\|_{L^{12}(S^3)} \leq C \|(-\Delta_{S^3} + (\lambda + i)^2) f\|_{L^{\frac{4}{3}}(S^3)}.$$

If we let  $f$  be an arbitrary  $L^2$  normalized eigenfunction  $e_\lambda$  corresponding to eigenvalue  $\lambda$ , then we ought to have

$$(3.18) \quad \|e_\lambda\|_{L^{12}(S^3)} \lesssim \lambda \|e_\lambda\|_{L^{\frac{4}{3}}(S^3)}.$$

A further testing with spherical harmonics and zonal functions indicate that the inequalities above are not saturated, which casts some doubt on the sharpness of the admissible range for  $(p, q)$ . But right now we are not able to prove, nor disprove the sharpness of that range even for round spheres. As Sogge pointed out to us that such difficulties were encountered during his study on Bochner-Riesz means in [7], [9], which may indicate that to prove or disprove the sharpness of the admissible range in our problem can be substantially difficult.

#### 4 non-positive curvature manifolds

Recall that in [1] we proved the following result

$$(4.1) \quad \left\| \sum_{|\lambda_j - \lambda| < 1/\log \lambda} E_j f \right\|_{L^{\frac{2n}{n-2}}(M)} \leq C(\log \lambda)^{-1} \lambda \|f\|_{L^{\frac{2n}{n+2}}(M)}, \quad \lambda \gg 1,$$

which is a special case of a recent unpublished estimate of Hassell and Tacey, also an  $L^p$  invariant of earlier bounds of Bérard [2]. Now we are going to invoke the complete version of this result to prove that:

**Theorem 4.1.** *If  $(M, g)$  is a compact manifold of dimension  $n \geq 3$  with nonpositive sectional curvature then we have, when  $p < \frac{2(n+1)}{n+3}$*

$$(4.2) \quad \left\| \sum_{|\lambda_j - \lambda| < 1/\log \lambda} E_j f \right\|_{L^2(M)} \leq C(\log \lambda)^{-\frac{1}{2}} \lambda^{\delta(p)} \|f\|_{L^p(M)}, \quad \lambda \gg 1.$$

By Theorem 1.3 this will immediately prove Theorem 1.4.

Before we go through the details of the proof, we want to point out that if we can prove (4.5) with  $\log \lambda$  replaced by  $\varepsilon \log \lambda$  in which  $\varepsilon$  is smaller than 1, and only depends on  $M$  and  $p$ , then  $L^2$  orthogonality would immediately show that (4.2) is proved with a larger constant  $C$  possibly depending on  $p$ . But certainly this is harmless for us.

So we need to prove (4.2) with  $\varepsilon \log \lambda$ , in which the specific value of  $\varepsilon$  is about to be determined later. Firstly, we claim that if we choose an even nonnegative function  $a \in \mathcal{S}(\mathbb{R})$  satisfying  $a(r) = 1$ ,  $|r| \leq 1/2$  and having its Fourier transform supported in  $(-1, 1)$ , then in order to (4.2) it is sufficient to prove that for the multiplier

$$(4.3) \quad a(\varepsilon \log \lambda(\lambda - P)) = \frac{1}{2\pi\varepsilon \log \lambda} \int \hat{a}(t/(\varepsilon \log \lambda)) e^{it\lambda} e^{-itP} dt$$

we have

$$(4.4) \quad \|a(\varepsilon \log \lambda(\lambda - P))f\|_{L^{p'}(M)} \leq (\varepsilon \log \lambda)^{-1} \lambda^{2\delta(p)} \|f\|_{L^p(M)}, \quad p < \frac{2(n+1)}{n+3}.$$

In fact, due to the non-negativity of  $a(r)$  especially  $a(r) \approx 1$  when  $r$  is near 0, we know that if we use the fact  $a = (\sqrt{a})^2$  then a  $TT^*$  argument and the above estimate will

immediately imply that

$$(4.5) \quad \left\| \sum_{|\lambda_j - \lambda| < 1/(\varepsilon \log \lambda)} E_j f \right\|_{L^2(M)} \leq C(\varepsilon \log \lambda)^{-\frac{1}{2}} \lambda^{\delta(p)} \|f\|_{L^p(M)}$$

due to  $L^2$  orthogonality.

We then proceed in the way that we proved the estimates on the local operator  $\mathfrak{S}_{loc}(P)$ , say breaking the  $t$  interval into one part when  $t \leq 1$  and the other one when  $1 \leq t \lesssim \log \lambda$  (c.f. (2.1) and (2.2)).

Now, if  $\psi \in C^\infty(\mathbb{R}^1)$  is an even function and  $\psi(r) = 1$  when  $|r| > 2$  and  $\psi(r) = 0$  then we claim that the operator  $a_\lambda^1(P)$  defined as

$$b_\lambda(P) = \frac{1}{2\pi\varepsilon \log \lambda} \int (1 - \psi(t)) \hat{a}\left(\frac{t}{\varepsilon \log \lambda}\right) e^{i(\lambda - P)t} dt$$

satisfies the estimate in (4.4) when  $p \leq 2(n+1)/(n+3)$ . This can be proved very easily by the  $L^p \rightarrow L^{p'}$  version of Lemma 3.2 and the fact that

$$|b_\lambda(\tau)| \leq C(\varepsilon \log \lambda)^{-1} (1 + |\lambda - \tau|)^{-N}.$$

So we need only to consider

$$(4.6) \quad \frac{1}{2\pi\varepsilon \log \lambda} \int (1 - \psi(t)) \hat{a}(t/(\varepsilon \log \lambda)) e^{i(\lambda - P)t} dt.$$

To proceed, we need to replace the  $e^{-itP}$  in the integrand above by  $\cos tP$  since we are going to use the latter's property relating the wave equation. Now, notice that since both  $\psi$  and  $a$  are even functions, the difference between the operator in the above formula and

$$\frac{1}{2\pi\varepsilon \log \lambda} \int \psi(t) \hat{a}(t/(\varepsilon \log \lambda)) e^{i\lambda t} \cos tP dt.$$

way is a smoothing operator with size of  $O(\lambda^{-N})$ , as  $P$  is a positive pseudodifferential operator. So we are reduced to prove that if we let  $a_\lambda(P)$  denote

$$a_\lambda(P) = \int \psi(t) \hat{a}(t/(\varepsilon \log \lambda)) e^{i\lambda t} \cos tP dt$$

then

$$(4.7) \quad \|a_\lambda(P)f\|_{L^{p'}(M)} \leq C\lambda^{2\delta(p)} \|f\|_{L^p(M)}, \quad p < 2(n+1)/(n+3).$$

We are going to use interpolation to prove (4.7). More specifically we want to prove that

$$(4.8) \quad \|a_\lambda(P)\|_{L^1 \rightarrow L^\infty} \leq C \log \lambda e^{c\varepsilon \log \lambda} \lambda^{(n-1)/2}$$

in which  $c$  is a small number depending on the geometry of the manifold  $M$  and

$$(4.9) \quad \|a_\lambda(P)\|_{L^2 \rightarrow L^2} \leq C \log \lambda.$$

The second one is obvious since we have the fact that  $\cos tP$  is a bounded  $L^2$  operator. The first one was already proved in [1] by using the fact that

$$\cos t\sqrt{-\Delta_g}(x, y) = \sum_{\gamma \in \Gamma} \cos t\sqrt{-\Delta_{\tilde{g}}}(x, \gamma y), \quad x, y \in D$$

in which  $D \subset \mathbb{R}^n$  is a fundamental domain of the universal covering map  $\Pi : \mathbb{R}^n \rightarrow M$ ,  $\Gamma$  is the fundamental group, or the deck transform group of  $M$ , and  $\tilde{g}$  is the pull-back of Riemannian metric by  $\Pi$ .

Now we just need to do the interpolation. By the fact that  $\log \lambda \in o(\lambda^\varepsilon)$  for an arbitrarily small  $\varepsilon > 0$ , after interpolation we have when  $p < 2(n+1)/(n+3)$  that there is a number

$$\varepsilon(p) = (n+1)\left(\frac{1}{p} - \frac{n+3}{2(n+1)}\right) > 0$$

so that

$$\|a_\lambda(P)\|_{L^{p'}(M)} \leq C\lambda^{2\delta(p)}\lambda^{\frac{2-p}{p}c\varepsilon_1-\varepsilon(p)}\|f\|_{L^p(M)} \leq C\lambda^{2\delta(p)}\|f\|_{L^p(M)}$$

if we choose  $\varepsilon_1$  small enough according to  $\varepsilon(p)$  and  $c$ . So (4.7) is proved. Notice that due to the appearance of  $\log \lambda$  we are not able to prove the end point estimate when  $p = 2(n+1)/(n+3)$ . In fact, by using Lemma 3.2 we showed in [1] that when  $p = 2(n+1)/(n+3)$  we can only obtain the following bound

$$\|a_\lambda f\|_{L^{\frac{2(n+1)}{n-3}}(M)} \leq C\lambda^{\frac{n-1}{n+1}} \log \lambda \|f\|_{L^{\frac{2(n+1)}{n+1}}(M)}$$

which is  $\log \lambda$  worse than the (4.7).

## 5 Torus $\mathbb{T}^n$

In this section we are going to prove Theorem 1.5 in a similar way to the non-positive curvature manifold case in the previous section. In fact, by an argument similar to the one prior to (4.7), we need only to study that if we define an operator for  $0 < \varepsilon(p) \leq 1$  as the following

$$(5.1) \quad a_\lambda(P) = \int \psi(t)\hat{a}(t/\lambda^{\varepsilon(p)})e^{it\lambda} \cos tP dt,$$

then whether we can have

$$(5.2) \quad \|a_\lambda(P)f\|_{L^{p'}(\mathbb{T}^n)} \leq \lambda^{2\delta(p)}\|f\|_{L^p(\mathbb{T}^n)}, p < \frac{2(n+1)}{n+3}.$$

Here as before  $\psi(t)$  is a smooth function with support outside  $(-4, 4)$  and equals to one  $|t| > 10$ , and  $\hat{a}(t)$  is supported in  $(-1, 1)$ . Both functions are even, similar to the non-positive curvature manifold case.

As we have seen in the proof to Theorem 1.3, if we can find such a value of  $\varepsilon(p)$  which satisfies (5.2), then any smaller positive number than  $\varepsilon(p)$  will do as well. Therefore in the following argument, we can simply focus on finding a largest possible  $\varepsilon(p)$  based on an interpolation argument similar to the one used in previous section.

Now we are going to prove the following estimates to interpolate with:

$$(5.3) \quad \|a_\lambda f\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{T}^n)} \leq C\lambda^{\frac{n-1}{n+1}}\lambda^{\varepsilon(p)}\|f\|_{L^{\frac{2(n+1)}{n+3}}(\mathbb{T}^n)},$$

and

$$(5.4) \quad \|a_\lambda f\|_{L^\infty(\mathbb{T}^n)} \leq C\lambda^{\frac{n-1}{2}}\lambda^{\varepsilon(p)\frac{n+1}{2}}\|f\|_{L^1(\mathbb{T}^n)}.$$

The first one is an easy application of Lemma 3.2 so we need only to prove the second one. In [1] we gave out the proof in  $n = 3$  case and the general dimensional case is similar. For the readers' convenience we brief it as following.



Recall that if we identify  $\mathbb{T}^n$  with its fundamental domain  $Q = (-\frac{1}{2}, \frac{1}{2}]^n$  in  $\mathbb{R}^n$ , then we have

$$(5.5) \quad \cos t \sqrt{-\Delta_{\mathbb{T}^n}}(x, y) = \sum_{j \in \mathbb{Z}^n} \cos t \sqrt{-\Delta_{\mathbb{R}^n}}(x - y + j), \quad x, y \in Q$$

then by the finite speed of propagation of  $\cos t P$  we are reduced in estimating the size of the following integral

$$(5.6) \quad \begin{aligned} A_1(x, y) &= \sum_{\substack{j \in \mathbb{Z}^n \\ |j| \leq \lambda^{\varepsilon(p)}}} \int \int e^{i(x-y+j) \cdot \xi} \psi(t) \hat{a}(t/\lambda^{\varepsilon(p)}) e^{it\lambda} \cos t |\xi| dt d\xi \\ &= \sum_{\substack{j \in \mathbb{Z}^n \\ |j| \leq \lambda^{\varepsilon(p)} \\ |x-y+j| \geq 1}} \int \int e^{i(x-y+j) \cdot \xi} \psi(t) \hat{a}(t/\lambda^{\varepsilon(p)}) e^{it\lambda} \cos t |\xi| dt d\xi \\ &\quad + \sum_{\substack{j \in \mathbb{Z}^n \\ |j| \leq \lambda^{\varepsilon(p)} \\ |x-y+j| < 1}} \int \int e^{i(x-y+j) \cdot \xi} \psi(t) \hat{a}(t/\lambda^{\varepsilon(p)}) e^{it\lambda} \cos t |\xi| dt d\xi \\ &= (I) + (II). \end{aligned}$$

Notice that  $(II)$  will disappear in odd dimension due to Huygens principle. Nonetheless it should not cause any harm in a general dimension neither due to the following simple argument. In fact, due to our choice on the fundamental domain  $Q$ , there are only  $O(2^n)$  many terms non-vanishing in  $(II)$ , so by Euler formula we need only to prove that

$$(5.7) \quad \int \int_{\mathbb{R}^n} e^{i(x-y+j) \cdot \xi} \psi(t) \hat{a}(t/\lambda^{\varepsilon(p)}) e^{it(\lambda \pm |\xi|)} d\xi dt \in O(\lambda^{-N}), \quad |x - y + j| < 1.$$

Integrating by parts with respect to  $t$  shows that we need only to prove (5.7) when an extra cut-off function  $\beta(|\xi|/\lambda)$  is inserted in the integrand for function  $\beta$  is as in . Then due to the fact that in the support of integrand we have  $|t| > 4$ , another integration by part in  $\xi$  variable completes the proof.

Now we notice that in  $(I)$  if we replace  $\psi(t)$  by  $1 - \psi(t)$  we will end up with  $O(2^n)$  many integrals like

$$\int_{\mathbb{R}^n} e^{i(x-y+j) \cdot \xi} (\Psi(\lambda - |\xi|) + \Psi(\lambda + |\xi|)) d\xi, \quad |x| > 1.$$

in which  $\Psi(r)$  is a Schwartz function. Now using polar coordinates  $\xi \rightarrow r\omega$  will immediately show that these integrals are in  $O(\lambda^{\frac{n-1}{2}})$ , which is better than (5.4). So we need only to prove

$$(5.8) \quad \sum_{\substack{j \in \mathbb{Z}^n \\ |j| \leq \lambda^{\varepsilon(p)} \\ |x-y+j| > 1}} \left| \int_{\mathbb{R}^n} e^{i(x-y+j) \cdot \xi} \lambda^{\varepsilon(p)} a(\lambda^{\varepsilon(p)}(\lambda \pm |\xi|)) d\xi \right| \leq \lambda^{\frac{n-1}{2} + \varepsilon(p) \frac{n+1}{2}}.$$

This can be proved immediately if again we use polar coordinates and the decay estimate of the Fourier transform of the spheres.

After proving (5.3) and (5.4) we can now do the interpolation. Let  $0 \leq t \leq 1$  is determined by the following equation

$$(5.9) \quad t + (1-t) \frac{n+3}{2(n+1)} = \frac{1}{p}, p \leq \frac{2(n+1)}{n+3},$$

then the interpolation shows that we have

$$(5.10) \quad \|a_\lambda f\|_{L^{p'}(\mathbb{T}^n)} \leq \lambda^{\frac{n-1}{n+1}(1-t)+\varepsilon(p)(1-t)+\frac{n-1}{2}t+\varepsilon(p)\frac{n+1}{2}t} \|f\|_{L^p(\mathbb{T}^n)}, p \leq \frac{2(n+1)}{n+3}.$$

So we need only to solve a positive  $\varepsilon(p)$  as the largest possible value which satisfies (5.2) when  $t > 0$  from the following equation

$$(5.11) \quad \varepsilon(p) = \frac{\frac{n(n-1)t+n-1}{2(n+1)} - \frac{n-1}{n+1}(1-t) - \frac{n-1}{2}t}{\frac{n+1}{2}t + 1 - t}, t = \frac{2(n+1)}{n-1} \frac{1}{p} - \frac{n+3}{n-1}.$$

An elementary derivative test shows that this function is increasing when  $t \in [0, 1]$  and  $\varepsilon(2(n+1)/(n+3)) = 0$ , which coincides with our interpolation as the  $(1, \infty)$  end has a better bound. In particular when  $p = 2n/(n+2)$  we have  $\varepsilon(p) = 1/(n+1)$ .

Now what we need is simply to compose the projections between  $L^p \rightarrow L^2$  and  $L^2 \rightarrow L^q$ . Here we just need to choose the weaker estimates during the composition, say for a general  $(1/p, 1/q)$  pair in the  $\overline{AA'}$  admissible range we just choose  $\varepsilon(p)$  when  $(1/p, 1/q)$  is below the line of duality, and  $\varepsilon(q')$  when it is above it. So not surprisingly the improvement is symmetric with respect to the line of duality. The closer it is, the better the improvement we can have.

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